Chapter 5

The transportation problem and the assignment problem

In this chapter we introduce the algorithms used to solve two specific linear problems: the transportation problem and the assignment problem.

5.1 The transportation problem

In the application of linear programming techniques, the transportation problem was probably one of the first significant problems studied. The problem can be expressed by the formulation of a linear model, and it can be solved using the simplex algorithm. However, and because of the special structure of the linear model, it can be solved with a more efficient method which is the topic of this chapter.

The transportation problem deals with the transportation of any product from m origins, $O_1, \ldots, O_m$, to n destinations, $D_1, \ldots, D_n$, with the aim of minimizing the total distribution cost, where:

- The origin O_i has a supply of a_i units, $i = 1, \ldots, m$.

- The destination D_j has a demand for b_j units to be delivered from the origins, $j = 1, \ldots, n$.

- c_{ij} is the cost per unit distributed from the origin O_i to the destination D_j, $i = 1, \ldots, m, \ j = 1, \ldots, n$.

In mathematical terms, the above problem can be expressed as finding a set of x_{ij}'s, $i = 1, \ldots, m$, $j = 1, \ldots, n$, to meet supply and demand requirements at a minimum distribution cost. The corresponding linear model is:

$$\min \ z \ = \ \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leq a_i, \ \ i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j, \ \ j = 1, \ldots, n$$

$$x_{ij} \geq 0, \ \ i = 1, \ldots, m, \ \ j = 1, \ldots, n$$

Thus, the problem is to determine x_{ij}, the number of units to be transported from O_i to D_j, so that supplies will be consumed and demands satisfied at an overall minimum cost.

The first m constraints correspond to the supply limits, and they express that the supply of commodity units available at each origin must not be exceeded. The next n constraints ensure that the commodity unit requirements at destinations will be satisfied. The decision variables are defined positive, since they represent the number of commodity units transported.

The transportation problem in standard form is shown below:

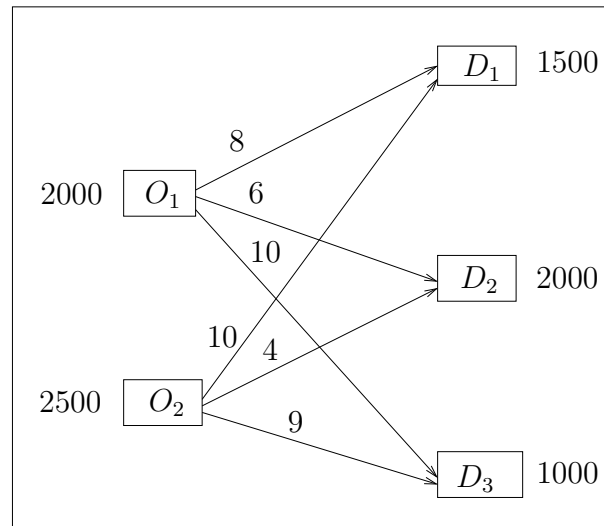$$\min \ z \ = \ \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \ \ i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \ \ j = 1, \ldots, n$$

$$x_{ij} \geq 0, \ \ i = 1, \ldots, m, \ \ j = 1, \ldots, n$$

Example. Two bread factories, O_1 and O_2, make the daily bread in a city. The bread is delivered to the three bakeries of the city: D_1, D_2 and D_3. The supplies

of bread factories, the demands of bakeries and the per unit transportation costs are displayed in the following graph:

We define the following decision variables:

x_{ij}: the number of loaves of bread to be distributed from the bread factory O_i to the bakery D_j, $i = 1, 2$, $j = 1, 2, 3$.

The corresponding linear model is the following:

$$\min \; z = 8x_{11} + 6x_{12} + 10x_{13} + 10x_{21} + 4x_{22} + 9x_{23}$$

subject to

$$
\begin{array}{rrrrrrl}
x_{11} & +x_{12} & +x_{13} & & & & = 2000 \\
& & & x_{21} & +x_{22} & +x_{23} & = 2500 \\
x_{11} & & & +x_{21} & & & = 1500 \\
& x_{12} & & & +x_{22} & & = 2000 \\
& & x_{13} & & & +x_{23} & = 1000
\end{array}
$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0$$

We can write the constraints in equation form, because the total supply is equal to the total demand. In order to analyze the structure of matrix $\mathbf{A}$, we write the model in matrix form.

$$\min z = (8\ ,\ 6\ ,\ 10\ ,\ 10\ ,\ 4\ ,\ 9) \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{pmatrix}$$

subject to

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{pmatrix} = \begin{pmatrix} 2000 \\ 2500 \\ 1500 \\ 2000 \\ 1000 \end{pmatrix}$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0$$

In this example, we have two origins, $m = 2$, and three destinations, $n = 3$. Matrix $\mathbf{A}$ has $2 + 3$ rows and 2×3 columns. It can be verified that the rank of the matrix is 4.

Note that each column of the matrix has only two 1's and 0's elsewhere. We use two subscripts to denote each one of the columns of matrix $\mathbf{A}$, as we did when we defined the decision variables for the transportation problem. Thus, in this example we denote by $\mathbf{a}_{11}$, $\mathbf{a}_{12}$, $\mathbf{a}_{13}$, $\mathbf{a}_{21}$, $\mathbf{a}_{22}$, $\mathbf{a}_{23}$ the six column vectors of the matrix. If we observe the positions at which the 1's can be found, we can easily see that, vector $\mathbf{a}_{11}$ for instance has a 1 in the 1st row and the other in the $m + 1$th row, vector $\mathbf{a}_{21}$ has the 1's in the 2nd and $m + 1$th rows and vector $\mathbf{a}_{23}$ has the 1's in the 2nd and $m + 3$th rows. In general, we can state that the two 1's of vector $\mathbf{a}_{ij}$ of matrix $\mathbf{A}$ are in rows i and $m + j$. $\qquad\square$

Therefore, the structure of matrix $\mathbf{A}$ depends on the number of origins and the number of destinations of the problem; any transportation problem having m

origins and n destinations has the same matrix $\mathbf{A}$. This matrix has $m + n$ rows and $m \times n$ columns. The rank of matrix $\mathbf{A}$ is $m + n - 1$, that is, any basis consists of $m + n - 1$ vectors. There are only two 1's in each column vector $\mathbf{a}_{ij}$ of matrix $\mathbf{A}$, and they are located in rows i and $m + j$, being 0 the rest of the values of the column vector. Thus, a transportation problem is specified by the number of origins, the number of destinations, the supplies, the demands and the per-unit transportation costs. All this information can be abbreviated in the form of a rectangular array.

5.2 The matrix format for the transportation problem

The relevant data for any transportation problem can be summarized in a matrix format using a tableau called *the transportation costs tableau* (see Figure 5.1). The tableau displays the origins with their supply, the destinations with their demand and the transportation per-unit costs.

	D_1	D_2	$\cdots$	D_n	Supply
O_1	c_{11}	c_{12}	$\cdots$	c_{1n}	a_1
O_2	c_{21}	c_{22}	$\cdots$	c_{2n}	a_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
O_m	c_{m1}	c_{m2}	$\cdots$	c_{mn}	a_m
Demand	b_1	b_2	$\cdots$	b_n	

Figure 5.1: The transportation costs tableau

Example. The transportation costs tableau of the example of page 152 is the following:

	D_1	D_2	D_3	Supply
O_1	8	6	10	2000
O_2	10	4	9	2500
Demand	1500	2000	1000	

The two origins of the problem are shown in the left column of the tableau, the bread factories O_1 and O_2, and their supply are displayed in the right column. The destinations appear at the top row of the array, bakeries D_1, D_2 and D_3, and their demand are at the bottom row of the array. Cells lying in the ith row and jth column of the tableau give the per-unit distribution cost coefficient c_{ij}, $i = 1, 2$, $j = 1, 2, 3$. $\qquad\square$

5.3 Formulating transportation problems

We defined the transportation problem as a problem of goods transportation. However, there is a wide range of different areas of application where it is very useful to formulate other problems by means of the matrix format for the transportation problem, and to apply the specific solution techniques that will be explained along this chapter.

In this section, we give two examples to show how to model an inventory planning problem and a production problem as transportation problems.

Example 1. An inventory planning problem.

A company needs to plan its production for the four quarters of the next year (one quarter is three months). It forecasts the following demands: 100 units the first quarter, 100 units the second, 200 units the third and 200 units the fourth. During each quarter, the production capacity is limited to 150 units. The demand of a quarter cannot be met in the next one. It costs 2 to produce a unit of product, which can be stored in a warehouse and be used to meet the demand of subsequent quarters. It costs 0.5 per quarter to store a unit of product in the warehouse.

In order to formulate this problem using the matrix format of the transportation problem, we define an array of four rows and four columns, each of which represents a quarter of the next year. The decision variables x_{ij} represent the number units produced in quarter i to meet the demand of quarter j, $i = 1, \ldots, 4$, $j = 1, \ldots, 4$.

- Supply of the four origins: 150, 150, 150, 150.

- Demand of the four destinations: 100, 100, 200, 200.

- If $i = j$, the per-unit production cost is $c_{ij} = 2$, $i, j = 1, \ldots, 4$.

- If $i < j$, the per-unit cost c_{ij} is computed as follows:

$$c_{ij} = \text{per-unit production cost} + \text{per-unit storage cost}.$$

For example, $c_{12} = 2.5$, $c_{13} = 3$, and the rest of the costs are computed in a similar way.

- If $i > j$, a penalty cost M is assigned to c_{ij} so that the variable x_{ij} is not basic, being M a large positive number.

This yields the following transportation costs tableau for the given inventory planning problem:

	1	2	3	4	Supply
1	2	2.5	3	3.5	150
2	M	2	2.5	3	150
3	M	M	2	2.5	150
4	M	M	M	2	150
Demand	200	150	200	100	

$\square$

Example 2. A production problem. A company manufactures a type of product in three different production plants: P_1, P_2 and P_3. Each of these production plants can produce up to 1500 units per month. The company supplies four customers who require each 1000, 1200, 1500 and 1000 units per month.

The company makes a benefit of 110 units for the manufacture of each unit of product. The unit transportation costs from each production plant to each customer are displayed below:

	1	2	3	4
P_1	30	10	25	20
P_2	15	25	30	10
P_3	20	30	15	20

We now proceed to formulate the matrix format of the transportation problem for the given production problem, so as to maximize the company's revenue:

- Supplies: $a_1 = 1500$, $a_2 = 1500$, $a_3 = 1500$.

- Demands: $b_1 = 1000$, $b_2 = 1200$, $b_3 = 1500$, $b_4 = 1000$.

- Each entry c_{ij} in the tableau represents the revenue obtained by manufacturing a unit of product in the production plant P_i to meet the demand of customer j, $i = 1, 2, 3$, $j = 1, 2, 3, 4$, and is computed like this:

$$c_{ij} = \text{per-unit production revenue} - \text{per-unit transportation cost}.$$

For instance,

$$c_{11} = 110 - 30 = 80, \quad c_{12} = 110 - 10 = 100, \quad c_{33} = 110 - 15 = 95.$$

The rest of the per-unit revenues are computed in a similar way.

The matrix format of the transportation problem to maximize the company's overall revenue is the following:

	1	2	3	4	Supply
P_1	80	100	85	90	1500
P_2	95	85	80	100	1500
P_3	90	80	95	90	1500
Demand	1000	1200	1500	1000	

□

5.4 Theorems and definitions

As we previously said, the transportation problem is just a special type of linear programming problem. We can take advantage of its special structure to adapt the simplex algorithm and to have a more efficient solution procedure. In this section we state some theorems and give some definitions that permit us derive the solution method for transportation problems.

Theorem 5.4.1 *The necessary and sufficient condition for a transportation problem to have a solution is that the total demand equals the total supply.*

Proof. According to the standard form of the transportation problem, each of the supplies a_i satisfies one of the following constraints:

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m$$

If we sum all the supplies, we get the total supply, which is:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \tag{5.1}$$

In a similar way, demands b_j satisfy the following constraints:

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n$$

The total demand is the following:

$$\sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j \tag{5.2}$$

The left-hand sides of equations (5.1) and (5.2) are equal. It follows that the equations mentioned are satisfied if and only if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

$\square$

The theorem above states that, under the assumption that the total supply equals the total demand, the transportation problem always has a feasible solution. However, such assumption is not always held. When the total supply does not equal the total demand, the problem has to be adapted before being solved, and the solution will be interpreted afterwards.

Definition 5.4.1 (Balanced problem.) *A transportation problem is said to be balanced if* $\displaystyle\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$

If the transportation problem is unbalanced, we have to convert it into a balanced one before solving it. There are two possible cases:

1.st case. The demand exceeds the supply, $\displaystyle\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j.$

It is not possible to satisfy the total demand with the existing supply. In this case, a dummy source or origin O_{m+1} is added to balance the model. Its corresponding supply and unit transportation cost are the following:

$$a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i.$$

$$c_{m+1,j} = 0, \ \ j = 1, \ldots, n.$$

As the supply a_{m+1} coming from the dummy origin O_{m+1} does not exist, any destination receiving transportation units from the dummy origin will experience shortage, which means that its demand will not be correctly satisfied. In some practical cases, it may be convenient to assign a unit transportation cost $c_{m+1,j}$ different from zero, a penalty of M for instance, in order to prevent shortage at a particular destination.

Example. Consider the following transportation problem in matrix format:

	1	2	3	Supply
1	2	4	3	10
2	6	1	4	20
Demand	20	20	20	

- Total supply $= a_1 + a_2 = 10 + 20 = 30$.

- Total demand $= b_1 + b_2 + b_3 = 20 + 20 + 20 = 60$.

The demand exceeds the supply. We add the dummy origin 3 to balance the problem, being $a_3 = 60 - 30 = 30$ its supply. We consider unit transportation costs c_{31}, c_{32} and c_{33} to be zero. This leads to the following balanced transportation problem in matrix format:

	1	2	3	Supply
1	2	4	3	10
2	6	1	4	20
3	0	0	0	30
Demand	20	20	20	

$\square$

2.nd case. The supply exceeds the demand. $\displaystyle\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$

Being the total supply higher than the total demand, we add a dummy destination D_{n+1} to the problem, such that its demand and unit transportation costs are:

$$b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j.$$

$$c_{i,n+1} = 0, \quad i = 1, \ldots, m.$$

The demand of the dummy destination is the difference between the total supply and the total demand. Unit transportation costs from any origin to the dummy destination are considered to be zero, because they do not correspond to real product transportations. However, it is possible to assign unit transportation costs different from zero if, for instance, we want to consider the cost of having product units stored.

Example. Consider the following transportation problem in matrix format.

	1	2	3	Supply
1	3	2	1	50
2	6	4	4	50
Demand	20	20	20	

- Total supply = $a_1 + a_2 = 50 + 50 = 100$.

- Total demand = $b_1 + b_2 + b_3 = 20 + 20 + 20 = 60$.

The total supply is higher than the total demand. We add a dummy destination 4, with a demand $b_4 = 100 - 60 = 40$. The unit transportation costs c_{14} and c_{24} are considered as zero. The balanced transportation problem in matrix format is:

	1	2	3	4	Supply
1	3	2	1	0	50
2	6	4	4	0	50
Demand	20	20	20	40	

□

Theorem 5.4.2 *A balanced transportation problem always has a feasible solution.*

Proof. Consider the linear model in standard form of a balanced transportation problem. Theorem 5.4.1 states that it has, indeed, a solution. Let us prove, now, that there exists a feasible solution. Consider:

$$T = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

It is easy to show that $x_{ij} = \frac{a_i b_j}{T}$ verifies the constraints, $i = 1, \ldots, m, \ j = 1, \ldots, n$, and is consequently a solution. Moreover, it is feasible, because $x_{ij} \geq 0$ holds, $i = 1, \ldots, m, \ j = 1, \ldots, n$. □

Theorem 5.4.3 *A balanced transportation problem always has a basic feasible solution. Such a solution consists of $m + n - 1$ positive variables at most.*

In the next section, we present two methods proposed to find a basic feasible solution for a transportation problem: *The Northwest Corner method* and *Vogel's approximation method*.

5.5 Finding an initial basic feasible solution

The procedure of calculating an initial basic feasible solution is performed in a tableau of the same dimensions as the transportation costs tableau; *the transportation solution tableau*, where each position (i, j) is associated with the decision variable x_{ij}, that is, the number of units of product to be transported from origin O_i to destination D_j (see Figure 5.2). Such positions (i, j) are called cells, and represent a solution. An empty cell denotes a value of zero.

	D_1	D_2	$\cdots$	D_n	Supply
O_1	x_{11}	x_{12}	$\cdots$	x_{1n}	a_1
O_2	x_{21}	x_{22}	$\cdots$	x_{2n}	a_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
O_m	x_{m1}	x_{m2}	$\cdots$	x_{mn}	a_m
Demand	b_1	b_2	$\cdots$	b_n	

Figure 5.2: The solution tableau for the transportation problem

5.5.1 The Northwest Corner method

Given a balanced transportation problem, and starting at a solution tableau with all the cells (i, j) empty, the following steps lead to an initial basic feasible solution.

* **Step 1**. In the rows and columns under consideration, select the cell (i, j) in the upper left-hand corner (northwest corner) of the solution tableau (to begin, $i = 1$, $j = 1$).

* Step 2. Assign to the variable x_{ij} the maximum feasible amount consistent with the row and the column requirements of that cell, that is, the value:

$$x_{ij} = \min\{a_i, b_j\}.$$

At least one of the requirements, the supply or the demand, will then be met. Adjust the supply a_i and the demand b_j as follows:

- If a_i happens to be the minimum, then the supply of the origin O_i becomes zero, and the row i is eliminated from further consideration. The demand b_j is replaced by $b_j - a_i$.

- If b_j happens to be the minimum, then the demand of the destination D_j becomes zero, and the column j is eliminated from further consideration. The supply a_i is replaced by $a_i - b_j$.

- If $a_i = b_j$, then the adjusted values for the supply a_i and the demand b_j become both zero. The row i and the column j are eliminated from further consideration.

* Step 3. Two cases may arise:

- If only one row or only one column remains under consideration, then any remaining cells (i, j), that is, variables x_{ij} associated with these cells, are selected and the remaining supplies are assigned to them. Stop.

- Otherwise, go to Step 1.

Example. Consider the balanced transportation costs tableau of the example on page 155.

	D_1	D_2	D_3	Supply
O_1	8	6	10	2000
O_2	10	4	9	2500
Demand	1500	2000	1000	

Supply $= 2000 + 2500 = 1500 + 2000 + 1000 =$ Demand

First iteration.

* **Step 1.** We select the upper left-hand corner cell $(1, 1)$, which corresponds to the 1st row and the 1st column of the solution tableau (see the symbol $*$ entered in the tableau shown below).

	D_1	D_2	D_3	Supply
O_1	$*$			2000
O_2				2500
Demand	1500	2000	1000	

* **Step 2.** Assign to the variable x_{11} the maximum feasible amount consistent with the row and the column requirements, and adjust the supply and the demand requirements.

$x_{11} = \min\{a_1, b_1\} = \min\{2000, 1500\} = 1500$

The adjusted supply of the origin O_1: $a_1 = 2000 - x_{11} = 500$.

The adjusted demand of the destination D_1: $b_1 = 1500 - x_{11} = 0$.

The demand of the destination D_1 has been satisfied; the column 1 is eliminated from further consideration.

	D_1	D_2	D_3	Supply
O_1	1500	$*$		~~2000~~ 500
O_2				2500
Demand	~~1500~~	2000	1000	

* **Step 3.** More than one row or one column remain under consideration in the solution tableau. Therefore, we go to Step 1, and proceed with a new iteration of the algorithm.

Second iteration. As in the first iteration, in the rows and columns under consideration we select the cell in the upper left-hand corner. This time, we select the cell $(1, 2)$, which corresponds to the 1st row and the 2nd column of the solution tableau (see the symbol $*$ entered in the tableau shown above). $x_{12} = \min \{500, 2000\} = 500$. We adjust the supply and the demand requirements: $a_1 = 500 - x_{12} = 0$ and $b_2 = 2000 - 500 = 1500$. The supply of the origin O_1 becomes zero, and the row 1 is eliminated from further consideration. The adjusted solution tableau is the following:

	D_1	D_2	D_3	Supply
O_1	1500	500		2000 500
O_2				2500
Demand	1500	2000 1500	1000	

Only one row remains under consideration: the second, which corresponds to the origin O_2. Thus, the two remaining cells $(2, 2)$ and $(2, 3)$ are selected and the remaining supplies $x_{22} = 1500$ and $x_{23} = 1000$ are assigned to the variables associated with them. The algorithm stops; the tableau shows an initial solution to the problem.

	D_1	D_2	D_3	Supply
O_1	1500	500		2000
O_2		1500	1000	2500
Demand	1500	2000	1000	

The initial solution obtained is basic and feasible. $m + n - 1 = 2 + 3 - 1 = 4$ variables have been assigned a positive value; they are the basic variables.

- The solution.

$$x_{11} = 1500, \ x_{12} = 500, \ x_{13} = 0, \ x_{21} = 0, \ x_{22} = 1500, \ x_{23} = 1000.$$

- The total transportation cost.

$$z = (1500 \times 8) + (500 \times 6) + (1500 \times 4) + (1000 \times 9) = 30000. \square$$

The Northwest Corner method is a very simple procedure proposed to find a basic feasible solution for a transportation problem. However, the unit transportation costs play no role in this method, which simply selects the upper left-hand corner variable and assigns a value to it. We next present another method, which takes the unit transportation costs into account, and which usually results in a basic feasible solution close to the optimal solution.

5.5.2 Vogel's approximation method

The main difference between the Northwest Corner method and Vogel's approximation method lays in the criteria used in Step 1 to select a cell in the solution tableau. Instead of selecting the upper left-hand corner, Vogel's approximation method computes row differences and column differences to select a cell. The row difference and the column difference are defined as follows:

- RD_i = the arithmetic difference between the smallest and the next smallest unit cost c_{ij} which remain under consideration in the row i, $i = 1, \ldots, m$.

- CD_j = the arithmetic difference between the smallest and the next smallest unit cost c_{ij} which remain under consideration in the column j, $j = 1, \ldots, n$.

The row differences and the column differences are used to make a more convenient selection of a cell in the solution tableau; a selection based on the unit transportation costs. Given a balanced transportation problem, and starting at a solution tableau with all the cells (i, j) empty, Vogel's approximation method consists of the following steps, and leads to a reasonably good initial basic feasible solution:

* **Step 1**. Compute for each row i and each column j the differences RD_i and CD_j. Among the rows and columns still under consideration, find the one with the largest difference, and find in it the cell (i, j) with the smallest unit transportation cost c_{ij}.

* **Step 2**. Assign to the variable x_{ij} the maximum feasible amount consistent with the row and the column requirements of that cell, that is, the value:

$$x_{ij} = \min\{a_i, b_j\}.$$

At least one of the requirements, the supply or the demand, will then be met. Adjust the supply a_i and the demand b_j as follows:

- If a_i happens to be the minimum, then the supply of the origin O_i becomes zero, and the row i is eliminated from further consideration. The demand b_j is replaced by $b_j - a_i$.

- If b_j happens to be the minimum, then the demand of the destination D_j becomes zero, and the column j is eliminated from further consideration. The supply a_i is replaced by $a_i - b_j$.

- If $a_i = b_j$, then the adjusted values for the supply a_i and the demand b_j become both zero. The row i and the column j are eliminated from further consideration.

* **Step 3**. Two cases may arise:

- If only one row or only one column remains under consideration, then any remaining cells (i, j), that is, variables x_{ij} associated with these cells, are selected and the remaining supplies are assigned to them. Stop.

- Otherwise, go to Step 1.

Example. Consider the balanced transportation costs tableau of the example on page 155. Let us apply Vogel's approximation method to find an initial basic feasible solution.

First iteration.

* **Step 1**. We compute the row and column differences, RD_i and CD_j, in the transportation costs tableau, and find the row or the column with the largest difference: $\max\{2, 5, 2, 2, 1\} = 5$, so that the second row is chosen. The smallest unit transportation cost in this row is $\min\{10, 4, 9\} = 4$. Thus, cell $(2, 2)$ has been selected.

Transportation costs tableau

	D_1	D_2	D_3	Supply	RD_i
O_1	8	6	10	2000	2
O_2	10	(4)	9	2500	(5)
Deman.	1500	2000	1000		
CD_j	2	2	1		

Transportation solution tableau

	D_1	D_2	D_3	Supply
O_1				2000
O_2		*		2500
Deman.	1500	2000	1000	

* Step 2. Assign to the variable x_{22} the maximum feasible amount consistent with the row and the column requirements of cell $(2,2)$:

$$x_{22} = \min\{2500, 2000\} = 2000.$$

We adjust the supply a_2 and the demand b_2 in the transportation solution tableau. Since the demand of the destination D_2 becomes zero, column 2 is eliminated from further consideration.

Transportation costs tableau

	D_1	D_2	D_3	Supply
O_1	8	6	10	2000
O_2	10	4	9	2500
Deman.	1500	2000	1000	

Transportation solution tableau

	D_1	D_2	D_3	Supply
O_1				2000
O_2		2000		500
Deman.	1500	0	1000	

* Step 3. More than one row or one column remain under consideration in the solution tableau. Therefore, we go to Step 1, and proceed with a new iteration of the algorithm.

Second iteration.

Once again, we proceed to compute the row and column differences, RD_i and CD_j, for rows, columns and costs c_{ij} still under consideration in the transportation costs tableau. In this case, $RD_1 = CD_1 = 2$ have the same largest difference. The choice can be made randomly; we choose row 1, for instance. The smallest unit transportation cost in row 1 is $c_{11} = 8$, which leads us to choose cell $(1,1)$. We assign to the variable x_{11} the maximum feasible amount consistent with the

row and the column requirements of cell $(1, 1)$: $x_{11} = \min\{1500, 2000\} = 1500$. We adjust the supply a_1 and the demand b_1 in the transportation solution tableau. Since the demand of the destination D_1 becomes zero, column 1 is eliminated from further consideration.

Transportation costs tableau

	D_1	D_2	D_3	Supply	RD_i
O_1	8	6	10	2000	2
O_2	10	4	9	2500	1
Deman.	1500	2000	1000		
CD_j	2		1		

Transportation solution tableau

	D_1	D_2	D_3	Supply
O_1	1500			500
O_2		2000		500
Deman.	0	0	1000	

Only one column remains under consideration; the third one, which corresponds to the destination D_3. Thus, the two remaining cells $(1, 3)$ and $(2, 3)$ are selected and the remaining supplies are assigned to the variables associated with them: $x_{13} = 500$ and $x_{23} = 500$. The algorithm stops; the solution tableau shows an initial basic feasible solution to the problem.

Transportation costs tableau

	D_1	D_2	D_3	Supply
O_1	8	6	10	2000
O_2	10	4	9	2500
Deman.	1500	2000	1000	

Transportation solution tableau

	D_1	D_2	D_3	Supply
O_1	1500		500	2000
O_2		2000	500	2500
Deman.	1500	2000	1000	

- The solution.

$$x_{11} = 1500, \ x_{12} = 0, \ x_{13} = 500, \ x_{21} = 0, \ x_{22} = 2000, \ x_{23} = 500.$$

- The total transportation cost.

$$z = (8 \times 1500) + (10 \times 500) + (4 \times 2000) + (9 \times 500) = 29500.$$

This initial solution is better than the one found by applying the Northwest Corner method, because the total transportation cost is lower.   □

5.6 Improvement of a basic feasible solution

The dual transportation problem is used to find an improved basic feasible solution. Consider a balanced transportation problem:

$$\min\ z\ =\ \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n$$

$$x_{ij} \geq 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

We denote by $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ the dual variables, which leads to the following dual model:

$$\max\ G = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

subject to

$$u_i + v_j \leq c_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

$$u_i, v_j : \text{unrestricted}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

Example. Consider the linear model of the transportation problem of the example on page 152. Let us calculate the corresponding dual model.

$$\min\ z = 8x_{11} + 6x_{12} + 10x_{13} + 10x_{21} + 4x_{22} + 9x_{23}$$

subject to

$$
\begin{array}{llllll}
x_{11} & +x_{12} & +x_{13} & & & & = 2000 \\
 & & & x_{21} & +x_{22} & +x_{23} & = 2500 \\
x_{11} & & & +x_{21} & & & = 1500 \\
 & x_{12} & & & +x_{22} & & = 2000 \\
 & & x_{13} & & & +x_{23} & = 1000
\end{array}
$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{21}, x_{23} \geq 0$$

The dual variables are denoted by u_1, u_2, v_1, v_2 and v_3, which leads to the following dual model:

$$\max \ G = 2000u_1 + 2500u_2 + 1500v_1 + 2000v_2 + 1000v_3$$

subject to

$$
\begin{array}{llll}
u_1 & +v_1 & & \leq 8 \\
u_1 & & +v_2 & \leq 6 \\
u_1 & & +v_3 & \leq 10 \\
u_2 & +v_1 & & \leq 10 \\
u_2 & & +v_2 & \leq 4 \\
u_2 & & +v_3 & \leq 9
\end{array}
$$

$$u_i, v_j : \text{unrestricted}$$

$\square$

The method used to compute the optimal solution of a transportation problem is actually a direct adaptation of the simplex method. Starting at an initial basic feasible solution, the method computes another basic feasible solution with a lower objective value z (bear in mind that the objective of the transportation problem is to minimize). To do so, we need to move from one basis to another, and thus, determine an entering variable and a leaving variable. This change in the basis is made according to the criteria studied in the improvement theorem.

5.6.1 Selection of the entering vector

Note that the variables in the transportation model are denoted by x_{ij}, the unit transportation costs by c_{ij} and column vectors of matrix $\mathbf{A}$ by $\mathbf{a}_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

According to the improvement theorem, we need to compute the reduced cost coefficients $z_{ij} - c_{ij}$ in order to decide whether the solution in hand can be improved:

$$z_{ij} - c_{ij} = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{a}_{ij} - c_{ij}.$$

Since $\mathbf{c}_B^T \mathbf{B}^{-1}$ is the vector of dual variables,

$$\mathbf{c}_B^T \mathbf{B}^{-1} = (u_1, \ldots, u_m, v_1, \ldots, v_n)$$

then,

$$z_{ij} - c_{ij} = (u_1, \ldots, u_m, v_1, \ldots, v_n)\mathbf{a}_{ij} - c_{ij}.$$

Since $\mathbf{a}_{ij}$ consists of 1's and 0's, and the only two 1's are in the ith and in the $(m + j)$th positions, we get:

$$z_{ij} - c_{ij} = u_i + v_j - c_{ij}$$

Hence, the reduced cost coefficients $z_{ij} - c_{ij}$ are computed through the use of the dual variables. Since $z_{ij} - c_{ij} = 0$ holds for any basic x_{ij}, and since there are $m + n - 1$ basic variables, we have a system of $m + n - 1$ equations of the type: $u_i + v_j - c_{ij} = 0$. The solution to the system will enable us to know the values of the $m + n$ dual variables $u_1, \ldots, u_m, v_1, \ldots, v_n$. Such system may be solved by setting any dual variable to an arbitrary value.

Once the values of the dual variables are known, the reduced cost coefficients $z_{ij} - c_{ij}$ for the nonbasic variables x_{ij} are computed. Taking into account that the objective of the transportation problem is to minimize the total transportation cost, two cases may arise:

- If $z_{ij} - c_{ij} \leq 0$ for any $i = 1, \ldots, m$, $j = 1, \ldots, n$, then the solution is optimal.

- If there is one or more $z_{ij} - c_{ij} > 0$, then the solution may be improved, and the variable with the maximum reduced cost coefficient $z_{ij} - c_{ij}$ is selected to enter the basis.

Example. Consider the basic feasible solution computed by applying the northwest corner method on page 166.

Transportation costs tableau

	D_1	D_2	D_3	Supply
O_1	8	6	10	2000
O_2	10	4	9	2500
Deman.	1500	2000	1000	

Transportation solution tableau

	D_1	D_2	D_3	Supply
O_1	1500	500		2000
O_2		1500	1000	2500
Deman.	1500	2000	1000	

The basic variables are x_{11}, x_{12}, x_{22} and x_{23}. We compute the values of the dual variables u_1, u_2, v_1, v_2 and v_3, by solving the following system:

$$x_{11} \text{ is basic} \quad \Rightarrow \quad z_{11} - c_{11} = 0 \quad \Rightarrow \quad u_1 + v_1 - 8 = 0.$$
$$x_{12} \text{ is basic} \quad \Rightarrow \quad z_{12} - c_{12} = 0 \quad \Rightarrow \quad u_1 + v_2 - 6 = 0.$$
$$x_{22} \text{ is basic} \quad \Rightarrow \quad z_{22} - c_{22} = 0 \quad \Rightarrow \quad u_2 + v_2 - 4 = 0.$$
$$x_{23} \text{ is basic} \quad \Rightarrow \quad z_{23} - c_{23} = 0 \quad \Rightarrow \quad u_2 + v_3 - 9 = 0.$$

Since there are four equations and five unknowns, the system has an infinite number of solutions. Any of the variables can be assigned a value arbitrarily and the remaining variables will be solved in a straightforward way. This procedure leads to one of the infinite solutions to the dual variables. For instance, we can set $u_1 = 0$ and solve the system, which yields $v_1 = 8$, $v_2 = 6$, $u_2 = -2$ and $v_3 = 11$.

Now, we can determine the reduced cost coefficient $z_{ij} - c_{ij}$ for every nonbasic variable x_{ij}. In this case, the nonbasic variables are x_{13} and x_{21}.

- $z_{13} - c_{13} = u_1 + v_3 - c_{13} = 0 + 11 - 10 = 1 > 0.$

- $z_{21} - c_{21} = u_2 + v_1 - c_{21} = -2 + 8 - 10 = -4 < 0.$

The reduced cost coefficient $z_{13} - c_{13}$ is positive. Then, the nonbasic variable x_{13} is chosen to enter the basis replacing one of the four basic variables. This new basis will let us compute a better basic feasible solution. $\square$

5.6.2   Selection of the leaving vector

In order to identify the leaving vector in the current basis, we need to take into account the following:

1. The $m + n - 1$ basic cells corresponding to any basic solution never contain a cycle. In the previous example, the variables x_{11}, x_{12}, x_{22} and x_{23} do not contain a cycle.

2. The $m + n - 1$ basic variables together with the entering variable contain a unique cycle.

   Such a unique cycle can be found proceeding in the following way: The cell that corresponds to the entering variable is marked by the symbol $\uparrow$, to indicate that a positive value will be given to the entering variable. Then, we cross out all the rows and columns which contain only one basic cell, assuming the cell marked by the symbol $\uparrow$ to be basic. We first cross out the rows, for instance, and afterwards the columns, then we check the rows again and so on. At the end of this process, the basic cells that have not been crossed out as well as the cell marked with the symbol $\uparrow$ contain the unique cycle we are looking for.

Once the cycle has been determined, we need to assign a value to the entering variable. Moreover, one of the basic variables will have to leave the basis, and thus become zero. We do it by adjusting the entries in the cycle: since the entering variable (the one that corresponds to the cell marked by the symbol $\uparrow$) will be assigned a positive value, the basic cells in the cycle that are in the same row or in the same column are marked by the symbol $\downarrow$, to indicate that the value of the corresponding variable will decrease to compensate the supply and the demand (bear in mind that the new solution has to continue satisfying the supply and the demand constraints). We carry on with the process of marking cells, so that the basic cells in the cycle that are in the same row or in the same column as the ones previously marked with the symbol $\downarrow$ are marked with the symbol $\uparrow$. The process ends when all the cells in the cycle are marked.

The first basic variable in the cycle decreased to zero among those marked by the symbol $\downarrow$ becomes the leaving variable. All the basic variables in the cycle are recomputed. The basic variables that are crossed out, the ones that do not belong to the cycle, remain unchanged. The entering variable is assigned the value the leaving variable had before being adjusted. As a consequence, we have a new basic feasible solution.

Example. We continue with the example on page 173. As we previously said, variable x_{13} was selected to enter the basis, and thus, a positive value will be assigned to it. We mark the cell $(1, 3)$ with the symbol $\uparrow$. We cross out column 1, because it contains only one basic cell: cell $(1, 1)$. We stop here the process of crossing out rows and columns, because all the rest have more than one basic cells, counting the entering cell $(1, 3)$ as being basic. Therefore, variables x_{12}, x_{13}, x_{22} and x_{23} constitute the cycle (see the four shaded cells in the tableau).

		D_1	D_2	D_3	Supply
O_1		1500	500 ↓		2000
O_2			1500 ↑	1000 ↓	2500
Demand		1500	2000	1000	

To compensate the increase in the value of variable x_{13}, the value of variables x_{12} and x_{23} in the same row or column, will be decreased. Next to them in the cycle is variable x_{22}, and thus, its value will be increased. The first basic variable in the cycle among those marked by the symbol ↓ to be decreased to zero is found by computing: $\min\{x_{12} = 500, x_{23} = 1000\} = 500$. Thus, x_{12} becomes the leaving variable.

Once the entering variable and the leaving variable are determined, all the variables in the cycle are recomputed; the new basic solution does not contain a cycle. Since the value of the leaving variable is 500, the variables in the cycle are updated by increasing or decreasing their value in 500 units. The new basic feasible solution is shown in the following tableau:

		D_1	D_2	D_3	Supply
O_1		1500		500	2000
O_2			2000	500	2500
Demand		1500	2000	1000	

We can check that the new solution satisfies the supply and demand constraints, just as the previous solution did.

- Solution.

$$x_{11} = 1500,\ x_{12} = 0,\ x_{13} = 500,\ x_{21} = 0,\ x_{22} = 2000,\ x_{23} = 500.$$

- Total transportation cost.

$$z = (8 \times 1500) + (10 \times 500) + (4 \times 2000) + (9 \times 500) = 29500.$$

We can verify that the solution has been improved because the total transportation cost is lower (see the transportation cost on page 167). □

5.7 The transportation tableau

Up to now, we have carried out all the calculations in two different tableaux: the transportation costs tableau and the transportation solution tableau. Moreover, we have computed the values u_i, v_j and $z_{ij} - c_{ij}$ outside the two tableaux.

Next, we present the *transportation tableau*, which puts together all the calculations needed to solve a transportation problem. Its structure is shown below:

	v_1		v_2		$\cdots$	v_n		
	$z_{11} - c_{11}$	c_{11}	$z_{12} - c_{12}$	c_{12}	$\cdots$	$z_{1n} - c_{1n}$	c_{1n}	
u_1	x_{11}		x_{12}			x_{1n}		a_1
	$z_{21} - c_{21}$	c_{21}	$z_{22} - c_{22}$	c_{22}	$\cdots$	$z_{2n} - c_{2n}$	c_{2n}	
u_2	x_{21}		x_{22}			x_{2n}		a_2
$\vdots$					$\ddots$			$\vdots$
	$z_{m1} - c_{m1}$	c_{m1}	$z_{m2} - c_{m2}$	c_{m2}	$\cdots$	$z_{mn} - c_{mn}$	c_{mn}	
u_m	x_{m1}		x_{m2}			x_{mn}		a_m
	b_1		b_2		$\cdots$	b_n		

Example. Let us continue with the previous example. Its corresponding transportation tableau is the following:

Leaves

$v_1 = 8$　　　$v_2 = 6$　　　$v_3 = 11$

Enters

	8		6	①	10	
$u_1 = 0$	1500		500	↓		2000
	−4	10		4		9
$u_2 = -2$			1500	↑	1000	↓	2500

　　　　　1500　　　2000　　　1000

　　　This transportation tableau shows all the information of the problem (see page 155), the initial basic feasible solution computed by the northwest corner method (see the solution tableau on page 166), and the values u_i, v_j and $z_{ij} - c_{ij}$ computed in Section 5.6 to select the entering variable (see page 174). Moreover, the tableau displays the cycle, the entering variable and the leaving variable (see tableau on page 176). □

5.8　The transportation algorithm

Consider a transportation problem which has as objective the minimization of the total transportation cost. The transportation algorithm consists of the following steps:

* **Step 1**. Balance the transportation problem.

* **Step 2**. Compute an initial basic feasible solution.

* **Step 3**. Compute the values $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ associated with the current basis.

* **Step 4**. Compute the reduced cost coefficient $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$ for every nonbasic variable x_{ij}.

  • If there is one or more $z_{ij} - c_{ij} > 0$, then the current solution may be improved. Select the variable with the most positive $z_{ij} - c_{ij}$ to enter the basis, and go to Step 5.

- If $z_{ij} - c_{ij} < 0$ for all nonbasic variables, then the current basic feasible solution is optimal and unique. Stop.

- If $z_{ij} - c_{ij} \leq 0$ for all nonbasic variables, and for one or more of the nonbasic variables $z_{ij} - c_{ij} = 0$ holds, then the optimal solution is multiple. Select anyone of those nonbasic variables to enter the basis, and go to Step 5.

* **Step 5**. Find the cycle which includes some of the current basic variables and the entering variable. Recompute the values of the variables in the cycle to obtain a new basic feasible solution. Go to Step 3.

5.9   An illustrative example

We shall now show how each of the steps of the transportation algorithm can be carried out directly in the transportation tableau. Consider the following transportation costs tableau:

	1	2	3	4	Supply
1	5	9	–	4	28
2	6	10	3	–	32
3	4	2	5	7	60
Demand	48	29	40	33	

The $-$ sign in cells $(1,3)$ and $(2,4)$ means that the transportation of product units is not possible between the corresponding origins and destinations. To ensure that variables x_{13} and x_{24} are not assigned any positive value, a large positive number M is assigned to the cost of the discarded transportations.

First iteration

Step 1. We balance the transportation problem. Supply $= 28 + 32 + 60 = 120$, Demand $= 48 + 29 + 40 + 33 = 150$. Thirty more units are needed to satisfy the total demand. Therefore, we add a dummy origin with a supply of 30 units and zero transportation unit costs. We place the information of the transportation problem in the transportation tableau.

5	9	M	4	
				28
6	10	3	M	
				32
4	2	5	7	
				60
0	0	0	0	
				30
48	29	40	33	

Step 2. We apply Vogel's approximation method to find an initial basic feasible solution. We compute the row and column differences, RD_i and CD_j, $i = 1, \ldots, 4$, $j = 1, \ldots, 4$. Columns 1 and 4 have the same largest difference. We select column 1, for instance. The smallest unit transportation cost in column 1 is in cell $(4, 1)$. We assign to variable x_{41} the maximum feasible amount consistent with the row and the column requirements: $\min\{30, 48\} = 30$ units. We adjust the supply and the demand, and eliminate row 4 from further consideration. This leads to the following tableau:

					RD_i
5	9	M	4	28	1
6	10	3	M	32	3
4	2	5	7	60	2
0	0	0	0	0	0
30					
18	29	40	33		
CD_j 4	2	3	4		

To proceed with the second iteration of Vogel's approximation algorithm, we recompute the row and column differences where needed. The largest is in column 2, $CD_2 = 7$, and the smallest unit transportation cost still under consideration in

column 2 is in cell $(3, 2)$. $\min\{60, 29\} = 29$ units are assigned to x_{32}, the supply and the demand are adjusted and column 2 is eliminated from further consideration, since the demand has become zero. The tableau is now as follows:

						RD_i
5		9		M		4	28	1
6		10		3		M	32	3
4		2	29	5		7	31	2
0	30	0		0		0	0	

18	0	40	33
CD_j 1	7	2	3

We repeat the process. We recompute RD_i and CD_j where needed, the largest are $RD_2 = CD_4 = 3$. We select row 2, for instance. The smallest cost still under consideration is $c_{23} = 3$, in cell $(2, 3)$. We assign $\min\{32, 40\} = 32$ to the variable x_{23}, adjust the supply and the demand, and eliminate row 2, because the supply has become zero.

						RD_i
5		9		M		4	28	1
6		10		3	32	M	0	3
4		2	29	5		7	31	1
0	30	0		0		0	0	

18	0	8	33
CD_j 1		2	3

Now, the largest difference is in column 3, $CD_3 = M - 5$. The smallest cost still under consideration in column 3 is $\min\{c_{13}, c_{33}\} = \min\{M, 5\} = 5$. Cell $(3, 3)$

is selected and $\min\{31, 8\} = 8$ units are assigned there. We adjust the supply and the demand, and eliminate column 3, because the demand has become zero.

	1	2	3	4		RD_i
1	5	9	M	4	28	1
2	6	10	3 (32)	M	0	0
3	4	2 (29)	5 (8)	7	23	1
4	0 (30)	0	0	0	0	0
	18	0	0	33		
CD_j	1		$M-5$	3		

There are still two rows and two columns under consideration. The largest differences are $RD_3 = CD_4 = 3$. We select row 3. The smallest cost still under consideration in row 3 is $\min\{c_{31}, c_{34}\} = \min\{4, 7\} = 4$. Cell $(3, 1)$ is selected and $\min\{23, 18\} = 18$ units are assigned there. We adjust the supply and the demand, and eliminate column 1.

	1	2	3	4		RD_i
1	5	9	M	4	28	1
2	6	10	3 (32)	M	0	0
3	4 (18)	2 (29)	5 (8)	7	5	3
4	0 (30)	0	0	0	0	0
	0	0	0	33		
CD_j	1			3		

Only one column remains under consideration; the fourth one. Thus, the two remaining cells $(1, 4)$ and $(3, 4)$ are selected and the corresponding variables are

assigned the remaining supplies: $x_{14} = 28$ and $x_{34} = 5$. Thus, Vogel's approximation method stops. The following tableau shows the initial basic feasible solution found.

5	9	M	4	
			28	28
6	10	3	M	
		32		32
4	2	5	7	
18	29	8	5	60
0	0	0	0	
30				30
48	29	40	33	

Step 3. Compute the values of the dual variables associated with the current basis. We set $u_3 = 0$. Since x_{31} is basic, $z_{31} - c_{31} = 0$ holds, and substituting the value $u_3 = 0$ into the equation $u_3 + v_1 - c_{31} = 0 + v_1 - 4 = 0$ we solve $v_1 = 4$. By substituting the variables already solved into the equations where they appear, we solve all the dual variables.

$$v_1 = 4 \quad v_2 = 2 \quad v_3 = 5 \quad v_4 = 7$$

	5	9	M	4	
$u_1 = -3$				28	28
	6	10	3	M	
$u_2 = -2$			32		32
	4	2	5	7	
$u_3 = 0$	18	29	8	5	60
	0	0	0	0	
$u_4 = -4$	30				30
	48	29	40	33	

Step 4. Compute the reduced cost coefficient $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$ for every nonbasic variable x_{ij}. For instance, $z_{21} - c_{21} = u_2 + v_1 - c_{21} = -2 + 4 - 6 = -4$. We proceed in a similar way to compute the rest of the $z_{ij} - c_{ij}$.
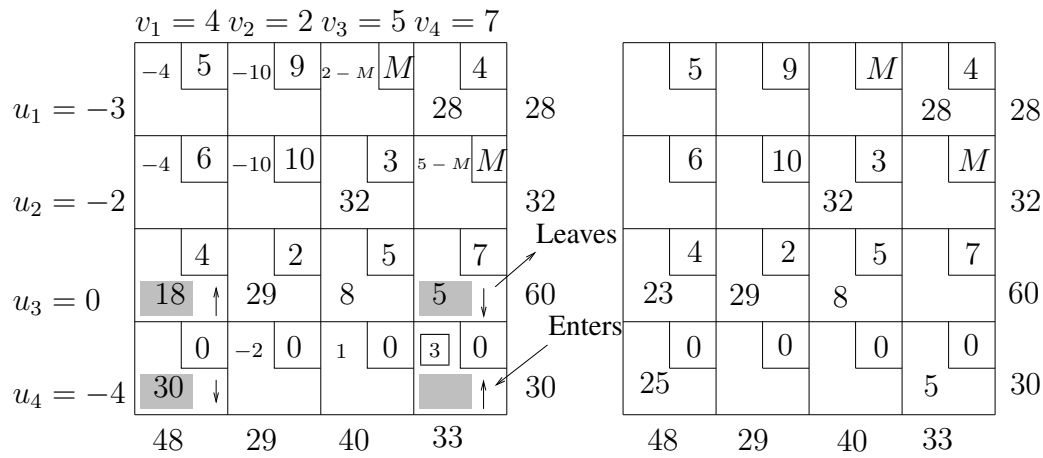
	$v_1 = 4$	$v_2 = 2$	$v_3 = 5$	$v_4 = 7$	
$u_1 = -3$	-4 | 5	-10 | 9	$2-M$ | M	4 28	28
$u_2 = -2$	-4 | 6	-10 | 10	3 32	$5-M$ | M	32
$u_3 = 0$	4 18	2 29	5 8	7 5	60
$u_4 = -4$	0 30	-2	0 | 1 | 0	$\boxed{3}$ | 0 Enters	30
	48	29	40	33	

As we can see, $z_{43} - c_{43} = 1$ and $z_{44} - c_{44} = 3$ are positive. We select the variable with the highest positive value, x_{44}, to enter the basis: cell $(4,4)$.

Step 5. The cycle includes the entering variable x_{44} and the current basic variables x_{31}, x_{34} and x_{41}. Increasing x_{44} requires decreasing x_{34} by the same amount to restore the demand of 33 in column 4, which in turn requires increasing x_{31} by this amount to restore the supply of 60 in row 3, which in turn requires decreasing x_{41} by this amount to restore the demand of 48 in column 1. The minimum among the decreasing variables: $\min\{5, 30\} = 5$. Thus, x_{34} becomes the leaving variable. We recompute the values of the variables in the cycle to obtain a new basic feasible solution, and continue the process in Step 3.

$v_1 = 4 \; v_2 = 2 \; v_3 = 5 \; v_4 = 7$

$u_1=-3$	$_{-4}$ 5	$_{-10}$ 9	$_{2-M}$ M	4 28	28
$u_2=-2$	$_{-4}$ 6	$_{-10}$ 10	3 32	$_{5-M}$ M	32
$u_3=0$	4 **18** ↑	2 29	5 8	7 **5** ↓	60
$u_4=-4$	0 **30** ↓	$_{-2}$ 0	$_{1}$ 0	$\boxed{3}$ 0 ↑	30
	48	29	40	33	

Leaves / Enters

	5	9	M	4 28	28
	6	10	3 32	M	32
	4 23	2 29	5 8	7	60
	0 25	0	0	0 5	30
	48	29	40	33	

Second iteration

The application of this procedure in a second iteration of the transportation algorithm (compute the dual variables and the $z_{ij} - c_{ij}$, identify the entering variable, find the cycle, identify the leaving variable, recompute the variables in the cycle), yields a better basic feasible solution, as it is shown below.

$v_1 = 4 \; v_2 = 2 \; v_3 = 5 \; v_4 = 4$

$u_1=0$	$_{-1}$ 5	$_{-7}$ 9	$_{5-M}$ M	4 28	28
$u_2=-2$	$_{-4}$ 6	$_{-10}$ 10	3 32	$_{2-M}$ M	32
$u_3=0$	4 **23** ↑	2 29	5 **8** ↓	$_{-3}$ 7	60
$u_4=-4$	0 **25** ↓	$_{-2}$ 0	$\boxed{1}$ 0 ↑	0 5	30
	48	29	40	33	

Leaves / Enters

	5	9	M	4 28	28
	6	10	3 32	M	32
	4 31	2 29	5	7	60
	0 17	0	0 8	0 5	30
	48	29	40	33	
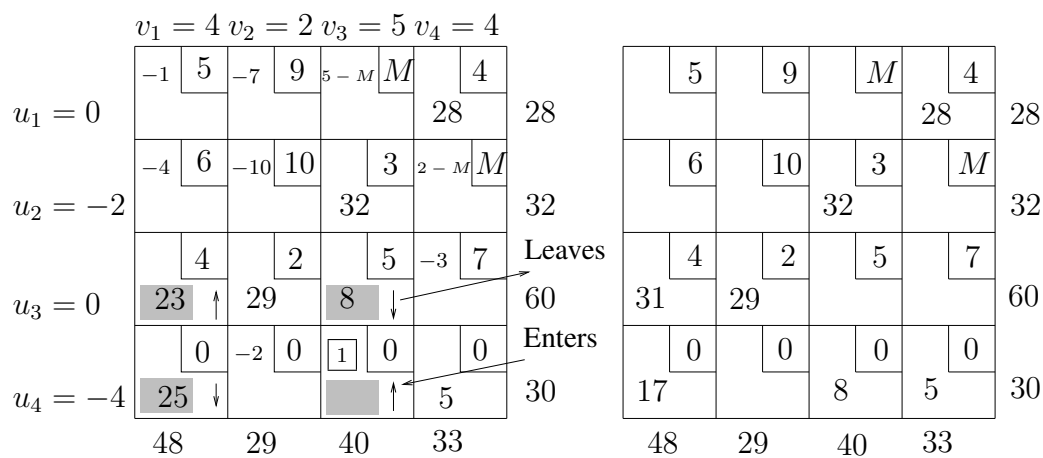
Third iteration.

We need to proceed with a third iteration of the transportation algorithm. We compute the dual variables and the reduced cost coefficients $z_{ij} - c_{ij}$. However, we see that $z_{ij} - c_{ij} < 0$ holds for all nonbasic variables, which means that the current basic feasible solution is optimal and unique.

$$v_1 = 0 \quad v_2 = -2 \quad v_3 = 0 \quad v_4 = 0$$

	$v_1=0$	$v_2=-2$	$v_3=0$	$v_4=0$	
$u_1 = 4$	$-1 \mid 5$	$-7 \mid 9$	$4-M \mid M$	4 ; 28	28
$u_2 = 3$	$-3 \mid 6$	$-9 \mid 10$	3 ; 32	$3-M \mid M$	32
$u_3 = 4$	4 ; 31	2 ; 29	$-1 \mid 5$	$-3 \mid 7$	60
$u_4 = 0$	0 ; 17	$-2 \mid 0$	0 ; 8	0 ; 5	30
	48	29	40	33	

- The optimal solution: $x_{14}^* = 28$, $x_{23}^* = 32$, $x_{31}^* = 31$, $x_{32}^* = 29$, $x_{41}^* = 17$, $x_{43}^* = 8$, $x_{44}^* = 5$. The first destination receives 17 dummy product units, the third 8 units and the fourth 5 units. Therefore, their demand has not been successfully satisfied because there was not enough supply.

- The minimum transportation cost:

$$z^* = (4{\times}28)+(3{\times}32)+(4{\times}31)+(2{\times}29)+(0{\times}17)+(0{\times}8)+(0{\times}5) = 390$$

$\square$

5.9.1 Degeneracy

If a solution to a given balanced transportation problem with m origins and n destinations has less than $m + n - 1$ positive variables, that is, if at least one of the basic variables has the value zero, then it is said to be *degenerate*. Degeneracy may occur in the following two cases:

- While computing an initial basic feasible solution, either by applying Vogel's approximation method or using the northwest corner method. If at Step 2 of either algorithm we find that when assigning to a variable the maximum feasible amount consistent with the row and the column requirements, both the supply and the demand are equal, then they become both

zero. This implies that a row and a column are eliminated at the same time from further consideration.

- While applying the transportation algorithm at Step 5, once we have identified the entering variable and found the cycle. If two decreasing variables in the cycle (two cells marked by the symbol ↓) tie for the minimum, then after recomputing the values in the cycle, they both become zero. However, only one of them can be selected as the leaving variable.

If degeneracy is encountered, then it can be handled quite easily by writing an explicit zero in the cell corresponding to the zero-valued basic variable, and treating it in the usual way. This enables us to distinguish the zero-valued basic variable from the nonbasic ones.

Example. Taking the following transportation costs tableau as starting point, let us compute an initial basic feasible solution:

	1	2	3	Supply
1	3	2	1	15
2	1	2	3	10
3	2	3	1	14
Demand	10	6	12	

We balance the problem, apply Vogel's approximation method and obtain the initial basic feasible solution that follows:

3	2	1	0	
	4	11		15
1	2	3	0	
10				10
2	3	1	0	
	6	8		14
10	6	12	11	

We can see that there are 5 positive variables: x_{13}, x_{14}, x_{21}, x_{32} and x_{33}. Since there are $m + n - 1 = 6$ basic variables, the solution is degenerate. Thus, one of the zero-valued variables is basic, and we need to distinguish it from the nonbasic ones. The way we do it is by choosing a zero-valued variable such that no cycle will be formed among it and the five positive variables. In this case, any variable x_{11}, x_{22}, x_{23}, x_{24} or x_{31} may be chosen. However, neither x_{12} nor x_{34} can be chosen, since they would form a cycle. If we select $x_{22} = 0$, for instance, we obtain the following degenerate basic feasible solution:

	3		2		1		0	
				4		11		15
	1		2		3		0	
10		0						10
	2		3		1		0	
		6		8				14
10		6		12		11		

□

Example. Suppose that, at a certain point in the solution process of a transportation problem, we obtain the feasible solution shown below. We, then, compute the reduced cost coefficients $z_{ij} - c_{ij}$:

	$v_1 = 6$	$v_2 = 4$	$v_3 = 1$	$v_4 = 4$	
$u_1 = 0$	6	4 $\;-3$ 4 $\;-4$ 8			
	10	12			22
$u_2 = 5$	9 $\;2$	9	6 $\;0$ 9		
		10	8		18
$u_3 = 3$	3 $\;6$	-4 11	4	7	
			15	5	20
	10	22	23	5	

Variable x_{21} is selected to enter the basis, since $z_{21} - c_{21} = 9$ is the maximum among all $z_{ij} - c_{ij} > 0$. Variables x_{11}, x_{12}, x_{21} and x_{22} constitute the cycle.

We recompute the variables in the cycle, and see that two of them become zero: $x_{11} = 0$ and $x_{22} = 0$. Remember that, since all the different bases in a vector space have the same number of vectors, only one of them can be selected to leave the basis. Therefore, one of them must remain in the basis, x_{11} for instance, while the other is selected to leave the basis. This yields the degenerate solution shown in the tableau on the right hand side.

$v_1 = 6 \; v_2 = 4 \; v_3 = 1 \; v_4 = 4$

Left tableau:

	6		4	$_{-3}$ 4	$_{-4}$ 8	
$u_1 = 0$	10 ↓	12 ↑				22
Enters	$_9$ 2	9	$_6$	$_0$ 9		
$u_2 = 5$	↑	10 ↓	8			18
Leaves	$_3$ 6	$_{-4}$ 11	4	7		
$u_3 = 3$			15	5		20
	10	22	23	5		

Right tableau:

	6	4	4	8	
0	22				22
	2	10	6	9	
10		8			18
	6	11	4	7	
		15	5		20
10	22	23	5		

We should proceed with the algorithm until the optimal solution is reached. □

5.9.2 Multiple optimal solutions

Given the transportation costs tableau shown below, we now proceed to calculate all the optimal solutions.

	1	2	3	4	5	Supply
1	4	1	2	6	9	100
2	6	4	3	5	7	120
3	5	2	6	4	8	120
Demand	40	50	70	90	90	

Applying the transportation algorithm, we obtain the following solution:

	$v_1=4$	$v_2=1$	$v_3=2$	$v_4=3$	$v_5=6$	
$u_1=0$	4; 40	1; 20	2; 40	-3 | 6	-3 | 9	100
$u_2=1$	-1 | 6	-2 | 4	3; 30	-1 | 5	7; 90	120
$u_3=1$	0 | 5	2; 30	-3 | 6	4; 90	-1 | 8	120
	40	50	70	90	90	

Since $z_{ij} - c_{ij} \leq 0$ for all nonbasic variables, and $z_{31} - c_{31} = 0$, the problem has multiple optimal solutions.

- An optimal solution: $x_{11}^* = 40$, $x_{12}^* = 20$, $x_{13}^* = 40$, $x_{23}^* = 30$, $x_{25}^* = 90$, $x_{32}^* = 30$, $x_{34}^* = 90$.

- The minimum transportation cost:

$$z^* = (4 \times 40) + (1 \times 20) + (2 \times 40) + (3 \times 30) + (7 \times 90) + (2 \times 30) + (4 \times 90) = 1400.$$

In order to obtain another optimal solution, we select variable x_{31} to enter the basis, and x_{32} to leave it. We recompute the variables in the cycle, which leads to another optimal solution.

	4	1	2	6	9	
	10	50	40			100
	6	4	3	5	7	
			30		90	120
	5	2	6	4	8	
	30			90		120
	40	50	70	90	90	

- Another optimal solution: $x_{11}^* = 10$, $x_{12}^* = 50$, $x_{13}^* = 40$, $x_{23}^* = 30$, $x_{25}^* = 90$, $x_{31}^* = 30$, $x_{34}^* = 90$.

- The minimum transportation cost:

$$z^* = (4{\times}10){+}(1{\times}50){+}(2{\times}40){+}(3{\times}30){+}(7{\times}90){+}(5{\times}30){+}(4{\times}90) = 1400.$$

$\square$

5.10 The assignment problem

The assignment problem is a special case of the transportation problem. It deals with assigning n origins (workers, for instance) to n destinations (jobs or machines, for instance,) with the goal of determining the minimum cost assignment. Each origin must be assigned to one and only one destination, and each destination must have assigned one and only one origin. c_{ij} represents the cost of assigning the origin O_i to the destination D_j, $i, j = 1, \ldots, n$.

Decision variables are defined like this:

$$x_{ij} = \begin{cases} 1 & \text{if } O_i \text{ is assigned to } D_j \\ 0 & \text{otherwise} \end{cases}$$

A linear model in standard form for the assignment problem is given by the following:

$$\min \ z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, \ldots, n$$

$$x_{ij} = 0, 1, \quad i, j = 1, \ldots, n$$

The first n constraints ensure that each origin is assigned to one and only one destination; the next n constraints ensure that each destination is assigned to one and only one origin.

If the number of origins and destinations are not equal, the assignment problem is unbalanced. In order to balance it, we can always add as many dummy

origins or dummy destinations as necessary. The assignment costs of the dummy origins or dummy destinations are zero.

The relevant data for any assignment problem can be summarized in a matrix format using a tableau called *the assignment costs tableau*.

$$
\begin{array}{c|cccc}
 & D_1 & D_2 & \ldots & D_n \\
\hline
O_1 & c_{11} & c_{12} & \ldots & c_{1n} \\
O_1 & c_{21} & c_{22} & \ldots & c_{2n} \\
\vdots & \vdots & \ddots & & \vdots \\
O_n & c_{n1} & c_{n2} & \ldots & c_{nn}
\end{array}
$$

Figure 5.3: The assignment costs tableau

Evidently, the assignment problem can be solved using the transportation algorithm described in Section 5.8. However, a highly efficient special algorithm was developed for the assignment problem, which will be analyze in the next section.

5.10.1   The Hungarian method

The special algorithm developed to solve the assignment problem is based on the following two theorems.

Theorem 5.10.1 *Consider the objective function of an assignment problem:*

$$
z = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}.
$$

If a solution x_{ij} is optimal for the objective function z, then it is also optimal for the problem in which the objective function has been replaced by z':

$$
z' = \sum_{j=1}^{n} \sum_{i=1}^{n} c'_{ij} x_{ij},
$$

where $c'_{ij} = c_{ij} - u_i - v_j$, and u_i and v_j are constant, $i, j = 1, \ldots, n$.

Proof.

$$z' = \sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}x_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n}(c_{ij} - u_i - v_j)x_{ij} =$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} - \sum_{j=1}^{n}\sum_{i=1}^{n} u_i x_{ij} - \sum_{j=1}^{n}\sum_{i=1}^{n} v_j x_{ij} =$$

$$= z - \sum_{i=1}^{n} u_i \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} v_j \sum_{i=1}^{n} x_{ij} =$$

$$= z - \sum_{i=1}^{n} u_i - \sum_{j=1}^{n} v_j = z - k.$$

The new objective function z' differs from the original z by a constant k. Therefore, the optimal values of x_{ij} are the same in both cases. $\square$

Theorem 5.10.1 enables us to transform the original assignment costs tableau into an equivalent one, by subtracting u_i from row i and then subtracting v_j from column j, $i, j = 1, \ldots, n$. So the objective function decreases by the value subtracted, but the solution to the assignment problem remains unchanged, since any solution must have exactly one $x_{ij} = 1$ in each row and column.

Theorem 5.10.2 *Given $c_{ij} \geq 0$ and a solution x_{ij} to the assignment problem, if*
$$z = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} = 0 \text{ holds, then the values of } x_{ij} \text{ are optimal, } i,j = 1, \ldots, n.$$

Proof. Since $x_{ij} = 0, 1$ and $c_{ij} \geq 0$, the objective value is $z \geq 0$. If the objective value is minimum for some values of x_{ij}, that is $z = 0$, then they are an optimal solution to the assignment problem, $i, j = 1, \ldots, n$. $\square$

The solution of an assignment problem proceeds by transforming the assignment costs tableau to create zero assignment-costs (see Theorem 5.10.1). Then, if a feasible assignment is found in which all the x_{ij}'s that equal 1 have zero costs and thus $z = 0$, the assignment is an optimal solution to the problem (see Theorem 5.10.2).

As we have seen, if a constant is subtracted from each cost in a row or column, the optimal solution to the problem remains unchanged. The Hungarian method

is based on this result, and keeps on transforming the assignment costs tableau, until the created zero entries yield an optimal assignment.

The Hungarian method is based on a fundamental result of König, which says: *If the largest number of independent zeros that can be chosen is n, then n lines can be chosen that contain all of the zeros.*

Next, we present the Hungarian algorithm, usually used to solve minimization assignment problems.

5.10.2   The Hungarian algorithm

The algorithm is used to solve assignment problems in which the objective is to minimize the total assignment cost, and consists of the following steps:

* **Step 1.** Balance the assignment problem.

* **Step 2.** Create zero entries in the rows. For each row in the assignment costs tableau, subtract the row minimum u_i from each element in the row, $u_i = \min_j \{c_{ij}\}$. The new entries in the resulting tableau are $c'_{ij} = c_{ij} - u_i$, $i, j = 1, \ldots, n$.

* **Step 3.** Create zero entries in the columns. For each column in the resulting tableau, subtract the column minimum v_j from each element in the column, $v_j = \min_i \{c'_{ij}\}$. The new entries are $c''_{ij} = c'_{ij} - v_j$, $i, j = 1, \ldots, n$.

* **Step 4.** Choose independent zeros. Find the row or column with the smallest number of zero entries. Choose one of its zeros, and cross out all the zeros in the same row or column. Proceed to choose more zeros among the ones that have not been crossed out, starting at the row or column with the smallest number of them, until all zeros are either chosen or crossed out.

   * If n independent zeros have been chosen, an optimal solution is available. Stop.

   * If less than n independent zeros have been chosen, then go to Step 5.

* **Step 5.** Draw the minimum number of lines (horizontal or vertical) that are needed to cover all zeros in the tableau, proceeding like this:

   5.1 Mark with a cross any row where none of its zeros has been chosen.

5.2 Mark with a cross the columns corresponding to the crossed out zeros in the rows marked in Step 5.1.

5.3 Mark with a cross the rows corresponding to the chosen zeros of the columns marked in Step 5.2.

Repeat Steps 5.2 and 5.3, until there are no rows or columns that meet the condition.

Draw a line through any row not marked and any column marked; these lines cover all the zeros in the tableau. Go to Step 6.

* **Step 6.** Create new zeros. Find the smallest nonzero entry (call its value k) that is uncovered by the lines drawn in the previous step. Subtract k from each cost that lies in an uncovered row and add k to each cost that lies in a covered column. Go to Step 4.

Example. Four building companies have presented their projects to a competition called to build buildings A, B, C and D. Each builder must be assigned the construction of a building. The following tableau shows the time each building company needs to build each of the buildings. The objective is to assign the construction of a building to each building company so that the total building time is minimized.

	1	2	3	4
A	58	58	60	54
B	66	70	70	78
C	106	104	100	95
D	52	54	64	54

* **Step 1.** The assignment problem is balanced.

* **Step 2.** For each row, we subtract the row minimum, that is, 54, 66, 95 and 52, from each element in the row, respectively.

	1	2	3	4
A	4	4	6	0
B	0	4	4	12
C	11	9	5	0
D	0	2	12	2

* **Step 3.** For each column, we subtract the column minimum, that is, 0, 2, 4 and 0, from each element in the column, respectively.

	1	2	3	4
A	4	2	2	0
B	0	2	0	12
C	11	7	1	0
D	0	0	8	2

* **Step 4.** Choose independent zeros.

  - There is only one zero in the first row. It is in cell $(A, 4)$, and we choose it. We cross out the zero in cell $(C, 4)$, because it is in the same column as the zero just chosen. This leads to the following tableau:

	1	2	3	4
A	4	2	2	$\boxed{0}$
B	0	2	0	12
C	11	7	1	$\emptyset$
D	0	0	8	2

  There are two zeros left in the second row, none in the third and two in the fourth.

  - We now check the columns: there are two zeros left in the first and only one in the second. Therefore, we choose the zero in cell $(D, 2)$ and cross out the zero in cell $(D, 1)$, because it is in the same row. The tableau is now like this:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline A & 4 & 2 & 2 & \boxed{0} \\ B & 0 & 2 & 0 & 12 \\ C & 11 & 7 & 1 & \emptyset \\ D & \emptyset & \boxed{0} & 8 & 2 \end{array}$$

We continue checking the columns. There is only one zero left in the third column, cell $(B, 3)$; we choose it and cross out the zero in cell $(B, 1)$, because it is in the same row. This yields the following tableau:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline A & 4 & 2 & 2 & \boxed{0} \\ B & \emptyset & 2 & \boxed{0} & 12 \\ C & 11 & 7 & 1 & \emptyset \\ D & \emptyset & \boxed{0} & 8 & 2 \end{array}$$

Since less than 4 independent zeros have been chosen, the optimal solution is not available yet. We proceed to Step 5 of the algorithm.

* Step 5. We now draw the minimum number of lines to cover all the zeros.

5.1 We mark row C, because the only zero it contains has not been chosen.

5.2 There is a zero crossed out in row C. We mark its corresponding column, the fourth.

5.3 There is a zero chosen in the fourth column; it lies on the first row, and we mark it. We return to Step 5.2.

5.2 There is no zero crossed out in the first row.

The process of marking rows and columns has finished. We draw a line through any row not marked and any column marked with a cross (see rows and columns in shadow).

	1	2	3	4	
A	4	2	2	0	X
B	0̸	2	0	12	
C	11	7	1	0̸	X
D	0̸	0	8	2	
				X	

The three lines in shadow (two rows and one column) cover all the zeros in the tableau. We proceed to Step 6.

* **Step 6.** Create new zeros. The smallest uncovered entry is 1. We subtract 1 from each cost that lies in the uncovered rows (the first and the third) and add 1 to each cost that lies in the covered column (the fourth). The updated tableau is the following:

	1	2	3	4
A	3	1	1	0
B	0	2	0	13
C	10	6	0	0
D	0	0	8	3

We return to Step 4.

* **Step 4.** Choose independent zeros.

	1	2	3	4
A	3	1	1	0
B	0	2	0̸	13
C	10	6	0	0̸
D	0̸	0	8	3

Since 4 independent zeros have been chosen, an optimal solution is available, and thus, the algorithm stops.

- An optimal solution.

  $A \to 4$:   Builder 4 will build A.

  $B \to 1$:   Builder 1 will build B.

  $C \to 3$:   Builder 3 will build C.

  $D \to 2$:   Builder 2 will build D.

- The minimum assignment cost:

 $c_{A4} + c_{B1} + c_{C3} + c_{D2} = 54 + 66 + 100 + 54 = 274$

$\square$

5.10.3   Maximization problems

The Hungarian method can only be applied to assignment problems where the objective is to minimize. In the cases where the objective is given in maximization form, it has to be transformed to the minimization form.

$$\min(-z) = \sum_{i=1}^{n} \sum_{j=1}^{n} -c_{ij}x_{ij}$$

However, this transformation of the objective function causes the assignment costs to be negative. Theorem 5.10.2 can only be used if costs are nonnegative. One way to make all entries in the assignment tableau nonnegative is to choose the minimum among the negative entries ($-c_{kl} = \min\{ -c_{ij} \ / -c_{ij} < 0 \}$), and to subtract it from each cost in the tableau, $c'_{ij} = -c_{ij} + c_{kl}$. The new entries in the tableau are, hence, nonnegative, $c'_{ij} \geq 0$.
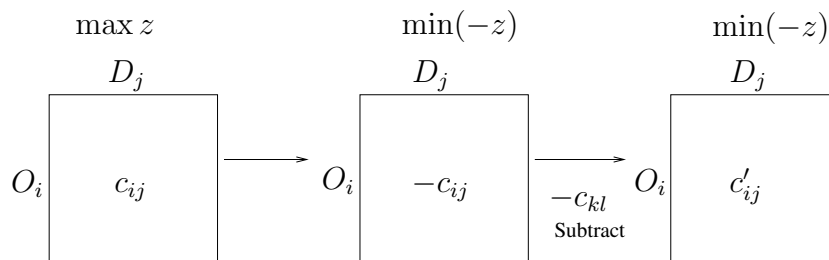
Example. An enterprise needs to employ three people for jobs A, B and C. Five people have applied for the jobs and are taking the selection tests: 1, 2, 3, 4

and 5. The following tableau shows the marks they obtained in the corresponding tests, graded from 1 to 10. The symbol $-$ in cell $(C, 4)$ means that person 4 is not suitable for job C.

	1	2	3	4	5
A	2	4	10	3	6
B	7	7	5	6	4
C	8	6	7	$-$	9

We need to find which person should be assigned to which job, so that the assignment is optimal. Since the entries in the tableau represent the mark obtained in each test, the objective of the problem is to maximize. Therefore, we transform it into the minimization form by replacing the entries c_{ij} with $-c_{ij}$. This yields an equivalent minimization assignment tableau.

	1	2	3	4	5
A	-2	-4	-10	-3	-6
B	-7	-7	-5	-6	-4
C	-8	-6	-7	$-$	-9

There are negative entries in the tableau; we subtract -10 from each entry in the tableau, and get the following:

	1	2	3	4	5
A	8	6	0	7	4
B	3	3	5	4	6
C	2	4	3	$-$	1

Now, there is no negative entry in the minimization tableau, and thus, the Hungarian method may be applied. On the other hand, to ensure that person 4 will not be assigned to job C, we place a sufficiently large positive number M in cell $(C, 4)$. The assignment costs tableau is now ready to apply the Hungarian algorithm.

$\square$

5.10.4 Multiple optimal solutions. Example

Consider the assignment costs tableau of the previous example.

	1	2	3	4	5
A	8	6	0	7	4
B	3	3	5	4	6
C	2	4	3	M	1

* **Step 1.** The problem is not balanced; we need to add two dummy origins
  (jobs) to balance it. Assignment costs corresponding to the dummy origins
  are zero.

	1	2	3	4	5
A	8	6	0	7	4
B	3	3	5	4	6
C	2	4	3	M	1
D	0	0	0	0	0
E	0	0	0	0	0

* **Step 2.** For each row, we subtract the row minimum, that is, 0, 3, 1, 0 and
  0, from each element in the row.

	1	2	3	4	5
A	8	6	0	7	4
B	0	0	2	1	3
C	1	3	2	M	0
D	0	0	0	0	0
E	0	0	0	0	0

* **Step 3.**   There are zero entries in all columns, and thus, subtracting the column minimum from each element in the column leaves the tableau as it is.

* **Step 4.** Choose independent zeros. Row 1 and row 3 contain only one zero entry. We start at row 1, for instance. We choose the zero in cell $(A, 3)$, and cross out all the zeros in the same row or column. We proceed to choose more zeros among the ones not crossed out, starting at the row or column with the smallest number of zeros not crossed out, until all zeros are either chosen or crossed out.

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
A & 8 & 6 & \boxed{0} & 7 & 4 \\
B & \boxed{0} & \emptyset & 2 & 1 & 3 \\
C & 1 & 3 & 2 & M & \boxed{0} \\
D & \emptyset & \boxed{0} & \emptyset & \emptyset & \emptyset \\
E & \emptyset & \emptyset & \emptyset & \boxed{0} & \emptyset \\
\end{array}
$$

Note that 5 independent zeros have been chosen, which means that an optimal solution is available. The algorithm stops.

- An optimal solution: $A \to 3$, $B \to 1$, $C \to 5$, $D \to 2$ and $E \to 4$.

  Person 2 and person 4 are assigned a dummy job, that is, they do not get any job.

- Assignment cost: $c_{A3} + c_{B1} + c_{C5} + c_{D2} + c_{E4} = 10 + 7 + 9 + 0 + 0 = 26$.

The independent selection of zeros in Step 4 could have been made in a different way. In fact, job B may be assigned either to person 1 (the previous solution) or to person 2. The assignment of job B to person 2 leads to another optimal solution, as shown below:

	1	2	3	4	5
A	8	6	[0]	7	4
B	$\emptyset$	[0]	2	1	3
C	1	3	2	M	[0]
D	[0]	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
E	$\emptyset$	$\emptyset$	$\emptyset$	[0]	$\emptyset$

- An optimal solution: $A \to 3$, $B \to 2$, $C \to 5$, $D \to 1$, $E \to 4$.

  In this case, person 1 and person 4 remain unemployed.

- Assignment cost: $c_{A3} + c_{B2} + c_{C5} + c_{D1} + c_{E4} = 10 + 7 + 9 + 0 + 0 = 26$.

There are other possible ways to choose independent zeros. The first optimal solution we found is $A \to 3$, $B \to 1$, $C \to 5$, $D \to 2$ and $E \to 4$, but we can also find this one: $A \to 3$, $B \to 1$, $C \to 5$, $D \to 4$ and $E \to 2$. The second optimal solution we found is $A \to 3$, $B \to 2$, $C \to 5$, $D \to 1$, $E \to 4$, but we can also find this one: $A \to 3$, $B \to 2$, $C \to 5$, $D \to 4$, $E \to 1$. Anyway, being D and E dummy jobs, the same people will remain unemployed. □